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Stopbands of the First-Order Bragg Interaction in a Parallel-Plate Waveguide Having Multiperiodic Wall Corrugations

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***Abstract*—The stopbands of the first-order Bragg interaction in a parallel-plate waveguide having multiperiodic wall undulations are investigated via the perturbation method of multiple scales. For a structure having two periods, the first-order Bragg interaction involves two as well as three coupled modes. Transition curves separating passbands from stopbands are found for all possible interactions. The effect of the multiple periodicity in the structure is found to be an increased band-width for the attenuation band as well as considerable attenuation throughout the band owing to the increased number of interactions. This is useful for the design of multichannel narrow-band microwave filters. The analysis is carried out for the first three dominant modes of the structure.**

I. INTRODUCTION

IN A REVIEW PAPER on wave propagation in periodic structures, Elachi [1] pointed out the need for investigating the stopbands of multiperiodic structures. In this paper, a first step in this direction is undertaken by studying the case of a doubly periodic structure for the stopbands of the first-order Bragg interaction. For this purpose we consider the propagation of TM modes in a parallel-plate waveguide having perfectly conducting walls that are perturbed in the direction of propagation according to the following wall distortion functions:

$$g_l(z)=\delta\sin k_l z, \qquad \text{at the lower plate} \tag{1}$$

$$g_u(z)=\delta\alpha\sin(k_u z+\theta), \qquad \text{at the upper plate} \tag{2}$$

where k_l and k_u describe the wavenumbers of the undulations of the lower and upper plates, respectively. Equations (1) and (2) describe wall undulations per unit of the separation d of the two plates so that δ is a dimensionless small parameter, much smaller than unity, and small enough for the Rayleigh hypothesis to hold [2]. The parameter α is a constant and θ is a constant phase angle.

The problem of a parallel-plate waveguide with one periodicity in the wall distortion function was treated by Nayfeh and Asfar [3], where the analysis was done for the first-order interaction of two propagating modes coupled by the wall perturbation. The second-order interaction of two modes in a periodic circular guide was analyzed by Asfar and Nayfeh [4]. For unbounded media, Chu and Tamir [5] analyzed mode coupling for the mth order Bragg interaction, while Jaggard and Elachi [6] considered the case of multiharmonic media where the different order Bragg interactions may add destructively to cause the disappearance of a stopband.

In the cases cited above [3]–[6], each Bragg interaction corresponds to a stopband except for the case of multiharmonic media where additional stopbands may appear [6]. It is the purpose of this paper to find the stopbands that appear in the case of bounded media having boundary perturbations. As discussed in the sequel, there are two sets of stopbands in a structure having two periods: the first corresponds to the coupling of two modes, and the second corresponds to the coupling of three modes. The analysis is made via the method of multiple scales [7]–[8]. The same approach was used by Nayfeh and Kandil [9] to

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derive interaction equations for three modes in a circular acoustic duct.

II. Formulation

The fields for a TM mode are derivable from a z-directed vector potential for waves propagating in the z-direction. We make all coordinates dimensionless by using the average separation of the plates d and the speed of light in the medium $c=(\mu\epsilon)^{-1/2}$ as reference quantities. Assuming a harmonic time variation of the form $\exp(-i\omega t)$, we obtain the following Helmholtz equation for the z-directed wave function ψ:

$$\nabla^2\psi+k^2\psi=0 \tag{3}$$

where

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}$$

and k is the dimensionless wavenumber for unbounded media. The boundary conditions on ψ are the vanishing of the tangential component of the electric field on the plates, which in terms of the potential function ψ give

$$\left(\frac{\partial^2}{\partial z^2}+k^2\right)\psi=-\delta k_l\cos(k_l z)\frac{\partial^2\psi}{\partial x\partial z}, \quad \text{at } x=\delta\sin(k_l z) \tag{4}$$

$$\left(\frac{\partial^2}{\partial z^2}+k^2\right)\psi=-\delta\alpha k_u\cos(k_u z+\theta)\frac{\partial^2\psi}{\partial x\partial z}, \quad \text{at } x=1+\delta\alpha\sin(k_u z+\theta). \tag{5}$$

III. Solution Using the Method of Multiple Scales

To apply the method of multiple scales to this problem, we introduce the scales $z_0=z$ and $z_1=\delta z$, where z_0 characterizes variations over distances of the order of a wavelength and z_1 characterizes the slow amplitude and phase modulations over distances large compared with the wavelength. Thus we transform the derivatives with respect to z as follows:

$$\frac{\partial}{\partial z}=\frac{\partial}{\partial z_0}+\delta\frac{\partial}{\partial z_1} \tag{6}$$

$$\frac{\partial^2}{\partial z^2}=\frac{\partial^2}{\partial z_0^2}+2\delta\frac{\partial^2}{\partial z_0\partial z_1}. \tag{7}$$

Then, we seek a perturbation expansion in powers of δ as follows:

$$\psi(x,z)=\psi_0(x,z_0,z_1)+\delta\psi_1(x,z_0,z_1). \tag{8}$$

Since we are interested in the first-order Bragg interaction, we carry the expansion to $0(\delta)$ only.

We substitute (6)–(8) into (3)–(5), expand ψ at the boundaries in Taylor series around $x=0$ and 1, equate coefficients of equal powers of δ on both sides of every equation, and obtain the following.

<u>0(1)</u>

$$\frac{\partial^2\psi_0}{\partial x^2}+\frac{\partial^2\psi_0}{\partial z_0^2}+k^2\psi_0=0 \tag{9}$$

$$\psi_0=0, \qquad \text{at } x=0 \text{ and } 1 \tag{10}$$

<u>0(δ)</u>

$$\frac{\partial^2\psi_1}{\partial x^2}+\frac{\partial^2\psi_1}{\partial z_0^2}+k^2\psi_1=-2\frac{\partial^2\psi_0}{\partial z_0\partial z_1} \tag{11}$$

$$\left(\frac{\partial^2}{\partial z_0^2}+k^2\right)\psi_1=-\sin(k_l z_0)\left(\frac{\partial^2}{\partial z_0^2}+k^2\right)\frac{\partial\psi_0}{\partial x} - k_l\cos(k_l z_0)\frac{\partial^2\psi_0}{\partial x\partial z_0} - 2\frac{\partial^2\psi_0}{\partial z_0\partial z_1}, \quad \text{at } x=0 \tag{12}$$

$$\left(\frac{\partial^2}{\partial z_0^2}+k^2\right)\psi_1=-\alpha\sin(k_u z_0+\theta)\left(\frac{\partial^2}{\partial z_0^2}+k^2\right)\frac{\partial\psi_0}{\partial x} - \alpha k_u\cos(k_u z_0+\theta)\cdot\frac{\partial^2\psi_0}{\partial x\partial z_0}-2\frac{\partial^2\psi_0}{\partial z_0\partial z_1}, \quad \text{at } x=1. \tag{13}$$

We seek a solution of (9) and (10) in the form of a linear combination of all possible modes (Rayleigh's hypothesis); that is,

$$\psi_0=\sum_{n=-\infty}^{\infty}A_n(z_1)\sin(n\pi x)\exp(ik_n z_0) \tag{14}$$

where n is an integer, $k_n^2=k^2-n^2\pi^2$, and the A_n are not determined at this level of approximation; they are determined from the solvability conditions of the first-order problem [8]. Substituting (14) into (11)–(13), we obtain

$$\frac{\partial^2\psi_1}{\partial z_0^2}+\frac{\partial^2\psi_1}{\partial x^2}+k^2\psi_1=-2i\sum_{n=-\infty}^{\infty}k_n A_n'\sin(n\pi x)\cdot\exp(ik_n z_0) \tag{15}$$

$$\left(\frac{\partial^2}{\partial z_0^2}+k^2\right)\psi_1=\frac{1}{2}i\sum_{n=-\infty}^{\infty}(n\pi)^3\cdot A_n\{\exp[i(k_n+k_l)z_0]-\exp[i(k_n-k_l)z_0]\}-\frac{1}{2}ik_l\sum_{n=-\infty}^{\infty}n\pi k_n A_n\{\exp[i(k_n+k_l)z_0]+\exp[i(k_n-k_l)z_0]\} \tag{16}$$

$$\left(\frac{\partial^2}{\partial z_0^2}+k^2\right)\psi_1=\frac{1}{2}i\alpha\sum_{n=-\infty}^{\infty}(-1)^n(n\pi)^3\cdot A_n\{\exp[i(k_n+k_u)z_0+i\theta]-\exp[i(k_n-k_u)z_0-i\theta]\}-\frac{1}{2}i\alpha k_u\sum_{n=-\infty}^{\infty}(-1)^n n\pi k_n A_n\cdot\{\exp[i(k_n+k_u)z_0+i\theta]+\exp[i(k_n-k_u)z_0-i\theta]\}. \tag{17}$$

Since the homogeneous parts of (15)–(17) are the same as (9) and (10), and since the latter equations have a nontrivial solution, the inhomogeneous equations (15)–(17) will have a solution if, and only if, solvability (integrability) conditions are satisfied [8]. To demonstrate this point, we consider the particular solution of (15)–(17) when $A'_n=0$; that is

$$\psi_1 \propto \sum_{n=-\infty}^{\infty} \{[\beta_{nl}^2 \sin(\beta_{nl})]^{-1} \exp[i(k_n+k_l)z_0] + [\beta_{nu}^2 \sin(\beta_{nu})]^{-1} \exp[i(k_n+k_u)z_0] + [\gamma_{nl}^2 \sin(\gamma_{nl})]^{-1} \cdot \exp[i(k_n-k_l)z_0] + [\gamma_{nu}^2 \sin(\gamma_{nu})]^{-1} \cdot \exp[i(k_n-k_u)z_0]\} \tag{18}$$

where

$$\beta_{nl}^2 = k^2-(k_n+k_l)^2 \qquad \beta_{nu}^2 = k^2-(k_n+k_u)^2 \tag{19a}$$

$$\gamma_{nl}^2 = k^2-(k_n-k_l)^2 \qquad \gamma_{nu}^2 = k^2-(k_n-k_u)^2. \tag{19b}$$

In analyzing the particular solution, we need to distinguish between two cases. First, all the $\beta_{nl}, \beta_{nu}, \gamma_{nl}, \gamma_{nu}$ are away from $m\pi$, where m is an integer. In this case, there exists a solution to the inhomogeneous equations (15)–(17) with (18) being its particular solution. Second, one or more of the β's and the γ's is equal to $m\pi$ which yields resonance frequencies defined by

$$k^2 = (k_n \pm k_{l,u})^2 + m^2\pi^2. \tag{20}$$

In this case, ψ_1 is unbounded and hence the inhomogeneous equations (15)–(17) do not have a solution. When one or more of the quantities $\beta - m\pi$ and $\gamma - m\pi$ is $0(\delta)$, ψ_1 becomes $0(\delta^{-1})$ and the supposedly small correction term $\delta\psi_1$ becomes the same order as ψ_0.

To determine a uniform solution for the resonant interactions, we must have $A'_n \neq 0$ and we must remove the resonant terms from the particular solution. To accomplish this we need to distinguish the following three possible types of interaction.

(i) The case of two modes coupled by either one of the wall corrugations in which case (20) is equivalent to the resonance conditions

$$k_{l,u} = k_m \mp k_p + \delta\sigma_1. \tag{21}$$

(ii) The case of three modes coupled by either one of the walls, that is, the resonance conditions are given by

$$k_{l,u} = k_m \mp k_p + \delta\sigma_1 \qquad k_{l,u} = k_s \mp k_p + \delta\sigma_2. \tag{22}$$

(iii) The case of three modes coupled by both walls with the resonance conditions

$$k_l = k_p \mp k_m + \delta\sigma_1 \qquad k_u = k_p \mp k_s + \delta\sigma_2. \tag{23}$$

The parameters σ_1 and σ_2 are called detuning parameters. They measure the nearness of the resonances. In general, they are different because they depend not only on the wavenumbers of the modes but also on the wall undulations. The minus sign in (21)–(23) corresponds to the interaction of modes traveling in the same direction whereas the plus sign corresponds to the interaction of oppositely directed modes. It can be shown that the results of the analysis for the latter can be obtained from those of the former simply by changing the signs of the wavenumbers of the backward modes.

Analysis for case (i) above follows the same lines of an earlier investigation [3] and is therefore omitted from the present analysis, only the pertinent results will be stated. Case (ii) is found to lead to a passband interaction only; that is, an interaction whereby the modes exchange their energies without attenuation. This is not of any importance in our case. Case (iii), however, is the most interesting since unlike case (i) a stopband occurs even for three modes traveling in the same direction. This is the subject of our concern in the next section.

IV. Stopbands of the Three-Mode Interaction

To find the solvability condition in this case we consider (23) with the minus signs; that is

$$k_m + k_l = k_p + \delta\sigma_1 \qquad k_s + k_u = k_p + \delta\sigma_2 \tag{24}$$

so that the terms leading to resonance in (16) and (17) may be expressed as

$$\exp[i(k_m+k_l)z_0] = \exp[i(k_p z_0 + \sigma_1 z_1)] \tag{25a}$$

$$\exp[i(k_s+k_u)z_0] = \exp[i(k_p z_0 + \sigma_2 z_1)] \tag{25b}$$

$$\exp[i(k_p-k_l)z_0] = \exp[i(k_m z_0 - \sigma_1 z_1)] \tag{25c}$$

$$\exp[i(k_p-k_u)z_0] = \exp[i(k_s z_0 - \sigma_2 z_1)]. \tag{25d}$$

Next, we seek a particular solution for ψ_1 in the form

$$\psi_1 = \sum_{n=m,p,s} f_n(x, z_1) \exp(ik_n z_0). \tag{26}$$

Substituting (26) into (15), we obtain

$$\frac{\partial^2 f_n}{\partial x^2} + n^2\pi^2 f_n = -2iA'_n k_n \sin(n\pi x), \qquad \text{for } n=m,p,s. \tag{27}$$

Multiplying both sides of (27) by $\sin(n\pi x)$ and integrating by parts from $x=0$ to $x=1$, we obtain

$$n\pi[f_n(0) - \cos(n\pi) f_n(1)] = -iA'_n k_n, \qquad \text{for } n=m,p,s. \tag{28}$$

The constants $f_n(0)$ and $f_n(1)$ are found from (16) and (17) upon substitution of (26). Thus (28) yields the following system of equations for the amplitudes of the interacting modes:

$$A'_m = \frac{p}{2mk_m}(k^2 - k_m k_p)\exp(-i\sigma_1 z_1)A_p \tag{29}$$

$$A'_p = \frac{-m}{2pk_p}(k^2-k_m k_p)\exp(i\sigma_1 z_1)A_m + \frac{\alpha s}{2pk_p}\cos(p\pi)\cos(s\pi) \cdot (k^2 - k_p k_s)\exp(i\sigma_2 z_1 + i\theta)A_s \tag{30}$$

$$A'_s = \frac{-\alpha p}{2sk_s}\cos(p\pi)\cos(s\pi)(k^2-k_p k_s) \cdot \exp(-i\sigma_2 z_1 - i\theta)A_p. \tag{31}$$

We note from (29)–(31) that the pth mode is directly coupled with the mth and sth modes, while the latter modes are indirectly coupled via the pth mode.

Equations (29)–(31) admit solutions of the form

$$A_m = a_m \exp[i(\lambda - \sigma_1)z_1] \quad (32a)$$

$$A_p = a_p \exp(i\lambda z_1) \quad (32b)$$

$$A_s = a_s \exp[i(\lambda - \sigma_2)z_1] \quad (32c)$$

provided that

$$\lambda^3 - (\sigma_2 + \sigma_1)\lambda^2 + \left[\sigma_1\sigma_2 - \frac{(k^2 - k_m k_p)^2}{4k_m k_p} - \alpha^2 \frac{(k^2 - k_p k_s)^2}{4k_p k_s}\right]\lambda + \frac{\sigma_2}{4k_m k_p}(k^2 - k_m k_p)^2 + \frac{\sigma_1 \alpha^2}{4k_p k_s}(k^2 - k_p k_s)^2 = 0. \quad (33)$$

If all the roots of (33) are real, there is a passband and the modes exchange their energies in accordance with energy conservation [3]; otherwise, there is a stopband. The significance of equation (33) is best discussed in connection with a numerical example.

V. Design of a Multichannel Narrow-Band Filter

We consider a waveguide section whose length is large compared with the plate separation d. For $d = 3$ cm, the first-, second-, and third-dominant waveguide modes having the wavenumbers k_1, k_2, and k_3, are cutoff at the frequencies 5, 10, and 15 GHz, respectively. In order to determine transition curves separating stopbands from passbands, we first determine the transition curves due to the case of two-mode interactions. The equation determining the characteristic exponent λ in this case can be abstracted from the interaction equations (29)–(31) by letting $A_s = 0$ in one case then $A_m = 0$ in the other case. The result would be

$$\lambda = \frac{1}{2}\left\{\sigma_1 \pm \left[\sigma_1^2 + \frac{(k^2 - k_m k_p)^2}{k_m k_p}\right]^{1/2}\right\} \quad (34a)$$

in the case of coupling of the modes p and m and

$$\lambda = \frac{1}{2}\left\{\sigma_2 \pm \left[\sigma_2^2 + \frac{\alpha^2(k^2 - k_p k_s)^2}{k_p k_s}\right]^{1/2}\right\} \quad (34b)$$

for the coupling of the modes p and s. It is clear from these relations that when the modes are opposite then λ is complex for a range of values of σ_1 and σ_2. This takes place when $k_p = -k_m$ or $-k_s$; that is, when an incident mode is reflected by the corrugations. In the present example there are two stopband interactions above each of the cutoff frequencies as shown in Fig. 1. The transition curve (solid line) occurring at a higher frequency in each case is the one corresponding to the interaction due to the larger wall wavenumber. The stopband lies to the left of the transition curves in each case and is bounded at the cutoff frequency. Inspection of the detuning shows that the interaction is strongest inside the stopband at frequencies where σ_1 or $\sigma_2 = 0$ to the left of the transition curve. Results are shown for corrugation periods $\Lambda_l \approx 8.66$ cm on the lower wall and $\Lambda_u \approx 3.1$ cm on the upper wall.

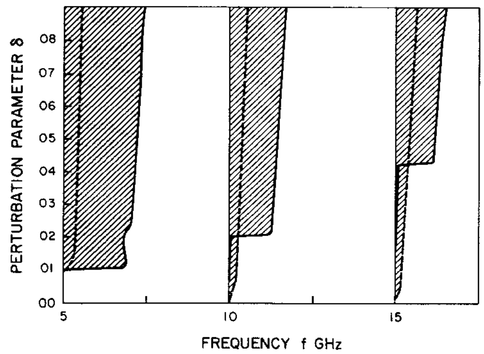

Fig. 1. Transition curves for two interacting modes. Shaded regions correspond to stopbands.

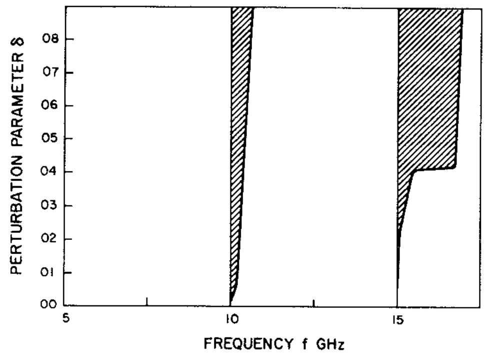

Fig. 2. Transition curves and stopbands for three interacting modes.

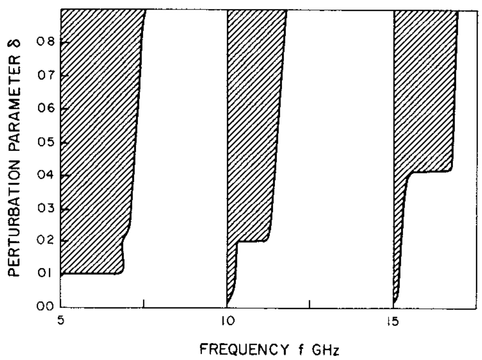

Fig. 3. Stopbands for the first-order Bragg interaction.

As for the three mode interactions, the present choice of k_l and k_u leads to two interactions out of the many possibilities that may be anticipated when either k_1 or k_2 or k_3 acts as the intermediate mode wavenumber that is directly coupled to the other two. The first of these is the interaction of k_2, $-k_2$, and k_1 with k_2 being the intermediate mode wavenumber above 10 GHz. The second interaction is more important and occurs above 15 GHz for an interaction of k_1, k_2, and k_3 with k_1 being the intermediate mode wavenumber. The stopbands of the three-mode interaction are shown in Fig. 2 as the shaded regions bounded by the cutoff frequency and the corre-

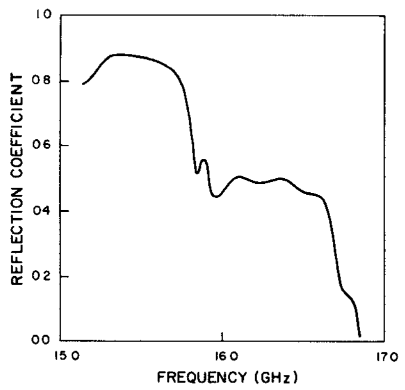

Fig. 4. Power reflection coefficient in the stopband above 15 GHz. Corrugation length 30 cm, $\delta=0.1$, and $\alpha=1$.

sponding transition curve. By combining the transition curves of the two types of interaction, we obtain the stopbands shown in Fig. 3.

The stopband above 15 GHz shows how the three-mode interaction, enhanced by the multiperiodicity of the structure, produces an attenuation band that is wider than that of the two-mode interaction. Moreover, there is a region where these two types of interaction overlap. In this region the attenuation is a maximum as seen in Fig. 4 [10] for a filter section 30 cm in length. This figure shows the behavior of the power reflection coefficient within the stopband above 15 GHz. One can identify two transitions in this diagram. The first corresponds to the transition curve of the two-mode interaction via the upper wall perturbation and carries with it a drop of about 35 percent in attenuation. This drop is due to the fact that the dispersion relation (33) always has a real root in the stopband so that part of the energy is transferred without attenuation. The second transition corresponds to the transition curve of the three-mode interaction into the passband.

We also note that the bandwidth may be increased by changing some of the parameters of the structure, such as the strength of the perturbation on the upper wall relative to that on the lower wall, or by decreasing the period of the upper wall. As an example, if $\delta=0.1$ and $\alpha=2$ then the stopband extends approximately up to 17.5 GHz as compared with 16.8 GHz for $\alpha=1$.

In conclusion, one can say that the effect of multiple periodicity is to produce a wide attenuation band that is characterized by considerable attenuation in regions where two-mode interactions coexist with three-mode interactions. For these reasons multiperiodic waveguides seem to be promising for narrow-band microwave filter applications.

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